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1976 J. Phys. A: Math. Gen. 9 L55

(http://iopscience.iop.org/0305-4470/9/6/003)

J. Phys. A: Math. Gen., Vol. 9, No. 6, 1976. Printed in Great Britain. © 1976

LETTER TO THE EDITOR

Correlated bond percolation on the Bethe lattice

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Received 11 March 1976

Abstract. A non-linear integral equation is derived for the percolation probability of the correlated bond percolation problem on the Bethe lattice. Analysis of the integral equation yields exact results for the critical percolation probability and the behaviour in the critical region.

1. Introduction

The exact solution to the random bond percolation problem on the Bethe lattice was first given by Fisher and Essam (1961). If the probability of a bond being present is p, then percolation (i.e. the existence of infinite chains of bonds) sets in for $p > p_c$, where $p_c = 1/\sigma$ for a Bethe lattice of coordination number $\sigma + 1$. Here we consider the solution to the correlated bond problem which was abstracted by Kirkpatrick (1973) from models of hopping conduction between localized states in semiconductors (Miller and Abrahams 1960, Ambegaokar *et al* 1971). In this problem, the probability that a bond connects two neighbouring sites of a lattice is no longer independent of the existence of other bonds leaving the sites.

2. The correlated bond model

The problem is as follows. Each site i of a Bethe lattice of coordination number $\sigma + 1$ is associated with a random variable E_i which is uniformly distributed on $[-1, 1]$. A bond joins neighbouring sites i and j provided

$$E_{ij} = \tfrac{1}{4}(|E_i| + |E_j| + |E_i - E_j|) \leqslant E \tag{1}$$

where $0 \leqslant E \leqslant 1$. What is the percolation probability (the probability that a given site is part of an infinite chain of sites linked by bonds) as a function of E?

It is helpful to illustrate the values of E_i, E_j (shown shaded in figure 1) which imply a bond between two neighbouring sites i and j (the diagram is for $E < \frac{1}{2}$). Only sites with $|E_i| \leqslant 2E$ can have bonds from them. We note immediately that the probability p that two neighbouring sites are joined by a bond irrespective of any other connections is equal to the shaded area divided by the total area of the square of allowed values of E_i and E_j,

$$p = 12E^2/4 = 3E^2 \qquad E \leqslant \tfrac{1}{2}. \tag{2}$$

If there were no correlations between bonds, the critical percolation probability would therefore be $1/\sigma$ and for $\sigma > 1$

$$E_c = 1/(3\sigma)^{1/2}. \tag{3}$$

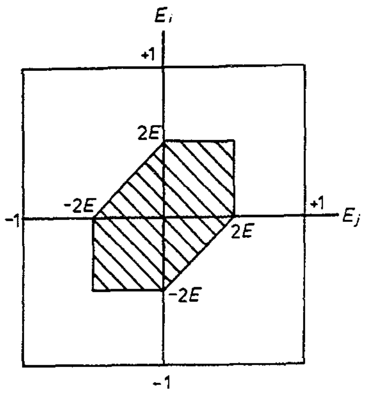

Figure 1. Neighbouring sites i and j have a bond joining them provided (E_j, E_i) is in the shaded region.

However correlations of a form which tend to cluster bonds together do exist. If a site i has a bond entering it, then $|E_i| \leqslant 2E$. Therefore the probability that i is also connected to a second neighbouring site is $\frac{3}{2}E$. A better estimate of the percolation threshold is given when this probability is equal to $1/\sigma$ which would imply

$$E_c = 2/3\sigma. \tag{4}$$

This condition is simply saying that for percolation to occur, a site which has a bond entering it must have open on average at least one of the σ other bonds leaving it. It will be seen that the exact E_c is only slightly less than this estimate.

3. Percolation probability

We now show how to obtain the percolation probability, $P(E)$, for the correlated bond model on the Bethe lattice. The method is analogous to that used by Essam (1972) to discuss random bond percolation.

Suppose the central site has associated parameter E_0. Since the $\sigma+1$ branches leaving this site are independent of each other if E_0 is fixed, we may write

$$P(E) = 1 - \tfrac{1}{2}\int_{-1}^{1} Q_E^{\sigma+1}(E_0)\,\mathrm{d}E_0. \tag{5}$$

Here, $Q_E(E_0)$ is the probability that no infinite chain leaves a site with parameter E_0 along a particular branch. The $\frac{1}{2}$ arises from the uniform probability distribution of E_0 on $[-1, 1]$.

Let $p_E(E_i, E_j) = 1 - q_E(E_i, E_j)$ be the probability that a bond connects neighbouring sites with parameters E_0, E_i. From (1),

$$p_E(E_i, E_j) = \theta(E - E_{ij}). \tag{6}$$

It is now apparent that $Q_E(E_0)$ satisfies the following non-linear integral equation:

$$Q_E(E_0) = \tfrac{1}{2}\int_{-1}^{1} q_E(E_0, E_i)\,\mathrm{d}E_i + \tfrac{1}{2}\int_{-1}^{1} p_E(E_0, E_i) Q_E^{\sigma}(E_i)\,\mathrm{d}E_i. \tag{7}$$

In (7) the first term is the probability that the bond in a given direction leaving E_0 is absent. The second term is the probability that this bond is present but that the σ remaining branches leaving E_i are all dead ends.

$Q_E(E_0)=1$ is always a solution of (7). However, we expect that there exists an E_c such that for $E>E_c$ there is another solution with $Q_E(E_0)<1$ (provided $|E_0|\leqslant 2E$) indicating that percolation can take place with a non-zero probability. From the discussion above, we expect $E_c<\frac{1}{2}$ (for $\sigma>1$) and all following expressions are only valid for $E\leqslant\frac{1}{2}$.

With the help of figure 1 and noting that $Q_E(E_0)$ is even in E_0, (7) may be written

$$Q_E(E_0)=\begin{cases}1-2E+\frac{1}{2}E_0+\int_0^{2E} Q_E^\sigma(E_i)\,\mathrm{d}E_i-\frac{1}{2}\int_{2E-E_0}^{2E} Q_E^\sigma(E_i)\,\mathrm{d}E_i & 0\leqslant E_0\leqslant 2E\\ 1 & E_0>2E\end{cases} \tag{8}$$

Let $x_E(\alpha)=1-Q_E(E_0/2E)$, then for $\alpha\leqslant 1$

$$x_E(\alpha)=2E\int_0^1 [1-(1-x_E(t))^\sigma]\,\mathrm{d}t-E\int_{1-\alpha}^1 [1-(1-x_E(t))^\sigma]\,\mathrm{d}t. \tag{9}$$

It follows from (9) that

(i) $x_E(\alpha)=0$ is a solution of (9).

(ii) Any non-zero solution of (9) is a monotonic decreasing function of α, since $\mathrm{d}x_E(\alpha)/\mathrm{d}\alpha<0$.

(iii) $x_E(0)=2x_E(1)$.

From (ii) and (iii) it follows that E_c is a well defined quantity in the sense that considered as a function of E for fixed α, the second solution required is such that $x_E(\alpha)\to 0$ as $E\to E_c^+$ for all α. We therefore make the *ansatz*

$$x_E(\alpha)\sim\delta^\nu(y(\alpha)+\delta z(\alpha)+\ldots). \tag{10}$$

Here $\delta=E-E_c$ and

$$y(\alpha)=\lim_{\delta\to 0^+} x_E(\alpha)/\delta^\nu\neq 0. \tag{11}$$

The exponent ν will later be shown to be 1. Substituting (10) in (9) gives a linear integral equation for $y(\alpha)$:

$$y(\alpha)=2E_c\sigma\int_0^1 y(t)\,\mathrm{d}t-E_c\sigma\int_{1-\alpha}^1 y(t)\,\mathrm{d}t. \tag{12}$$

It is easily seen that $y(\alpha)$ satisfies the second-order differential equation

$$\mathrm{d}^2y(\alpha)/\mathrm{d}\alpha^2=-X^2y(\alpha) \tag{13}$$

where $X=E_c\sigma$. The solution is of the form

$$y(\alpha)=A\sin X\alpha+B\cos X\alpha. \tag{14}$$

Substituting back in (12) we find

$$y(\alpha)=C(\cos X\alpha-\tfrac{1}{2}\sin X\alpha) \tag{15}$$

where C is a non-zero constant and X must satisfy the condition

$$(1-\sin X)/\cos X=\tfrac{1}{2}. \tag{16}$$

The relevant solution of (16) gives for the percolation threshold

$$E_c = \sin^{-1}(\tfrac{3}{5})/\sigma = 0{\cdot}6435/\sigma. \tag{17}$$

This exact result is to be compared with the value $E_c = 2/3\sigma$ obtained by simple arguments in § 1. The small difference arises when detailed account is taken of the variation with E_i of the conditional probability that further bonds leave a site i given that a bond enters i.

To obtain the exponent ν and the constant C it is necessary to make use of the non-linearity of (9). This may be done by solving for the next order term in the expansion (10). The function $z(\alpha)$ may be shown to satisfy the equation

$$z(\alpha) = \left(\int_0^1 - \tfrac{1}{2}\int_{1-\alpha}^1\right) \mathrm{d}t\,(2Xz(t) + 2\sigma y(t) - X(\sigma-1)y^2(t)) \tag{18}$$

where the term in $y^2(t)$ is only included if $\nu = 1$. This equation may again be converted into a differential equation but it turns out that only if $\nu = 1$ can a solution be found to (18). In this case

$$z(\alpha) = Dy(\alpha) - \sigma(\alpha - \tfrac{1}{2})y(1-\alpha) + \tfrac{5}{12}(\sigma-1)C^2 - \tfrac{1}{3}(\sigma-1)y(\alpha)(y(1-\alpha) + \tfrac{1}{2}y(\alpha)) \tag{19}$$

provided

$$C = \frac{15}{4}\frac{\sigma}{\sigma-1}. \tag{20}$$

D is a constant undetermined to this order.

The critical behaviour of the percolation probability can now be obtained from (5):

$$P(E) \sim \frac{15}{4}\frac{\sigma+1}{\sigma-1}(E - E_c) + O(E - E_c)^2. \tag{21}$$

Figure 2 shows the form of $P(E)$ for $\sigma = 4$.

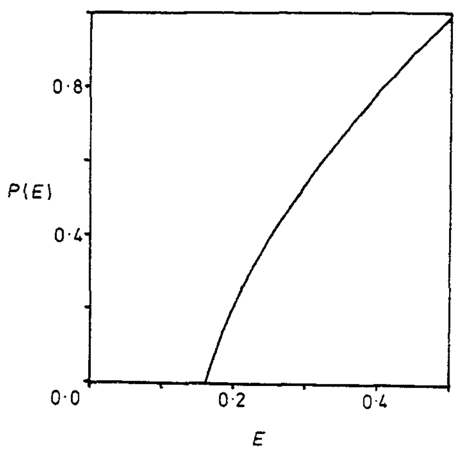

Figure 2. The percolation probability $P(E)$ for the correlated bond model on the Bethe lattice of coordination number $\sigma + 1 = 5$. The percolation threshold is at $E = 0{\cdot}1609$.

4. Conclusion

The solution of the correlated bond percolation problem on the Bethe lattice has been presented. The exact percolation threshold is given by equation (17) and the percolation probability is found to behave linearly in the critical region. It is intended to extend this work to the treatment of the electrical conductivity of a correlated resistor network.

Acknowledgments

I would like to thank Dr A Coniglio and Dr N Rivier for stimulating discussions about this work and Professor N H March for a critical reading of this letter. The award of an SRC postgraduate studentship is also gratefully acknowledged.

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